7.1 a)

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{\frac{k}{2}} = \sum_{k=1}^{\infty}\left(\left(\frac{1}{2}\right)^{\frac{1}{2}}\right)^k = \sum_{k=1}^{\infty}\sqrt{\frac{1}{2}}^{\,k} = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k$$

$$= \sum_{k=1}^{\infty} q^k = \sum_{k=0}^{\infty} q^k - 1 = \frac{1}{1-q} - 1 \quad \text{mit } q = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{1-\frac{1}{\sqrt{2}}} - 1 = \frac{\sqrt{2}}{\sqrt{2}-1} - 1 = \frac{\sqrt{2}}{\sqrt{2}-1} - \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1}$$

b)

$$\sum_{k=1}^{\infty}\frac{5^{2-k}}{2^k 3^{2k}} = \sum_{k=1}^{\infty}\frac{5^2}{2^k 5^k 9^k} = 25\sum_{k=1}^{\infty}\frac{1}{90^k} = 25\sum_{k=1}^{\infty}\left(\frac{1}{90}\right)^k$$

$$= 25\sum_{k=1}^{\infty} q^k = 25\left(\sum_{k=0}^{\infty} q^k - 1\right) = 25\left(\frac{1}{1-q} - 1\right) \quad \text{mit } q = \frac{1}{90}$$

$$= 25\left(\frac{1}{1-\frac{1}{90}} - 1\right) = 25\left(\frac{90}{90-1} - 1\right) = 25\left(\frac{90}{89} - \frac{89}{89}\right) = \frac{25}{89}$$

7.2 a)

$$\sum_{k=1}^{\infty}\frac{1}{k^3} = \sum_{k=1}^{\infty} a_k \quad \text{mit } a_k = \frac{1}{k^3} = f(k)$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty}\frac{1}{x^3}\,dx = \lim_{\mu\to\infty}\int_1^{\mu}\frac{1}{x^3}\,dx = \lim_{\mu\to\infty}\left[-\frac{1}{2}\frac{1}{x^2}\right]_1^{\mu}$$

$$= \lim_{\mu\to\infty}\left(-\frac{1}{2}\frac{1}{\mu^2} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow \text{Konvergent}$$

b)

$$a_k = \frac{1}{3^k} = \left(\frac{1}{3}\right)^k \quad \lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\left(\frac{1}{3}\right)^k} = \lim_{k\to\infty}\frac{1}{3} = \frac{1}{3} < 1$$

$\Rightarrow$ konvergent

c)

$$a_k = \frac{k^2}{k!} \quad \frac{a_{k+1}}{a_k} = \frac{\frac{(k+1)^2}{(k+1)!}}{\frac{k^2}{k!}} = \frac{(k+1)^2 k!}{k^2 (k+1)!} = \frac{(k+1)^2 k!}{k^2 k!(k+1)} = \frac{k+1}{k^2} = \frac{1}{k} + \frac{1}{k^2}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left(\frac{1}{k} + \frac{1}{k^2}\right) = 0 < 1 \Rightarrow \text{konvergent}$$

d)

$$a_k = \left(1+\frac{1}{k}\right)^{k^2} = \left(1+\frac{1}{k}\right)^{k\cdot k} = \left(\left(1+\frac{1}{k}\right)^k\right)^k$$

$$\sqrt[k]{|a_k|} = \sqrt[k]{\left(\left(1+\frac{1}{k}\right)^k\right)^k} = \left(1+\frac{1}{k}\right)^k$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e > 1 \Rightarrow \text{divergent}$$

e)

$$a_k = \frac{k}{2^k} \qquad \frac{a_{k+1}}{a_k} = \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \frac{(k+1)2^k}{k\,2^{k+1}} = \frac{(k+1)2^k}{k2^k\cdot 2} = \frac{k+1}{2k}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{k+1}{2k} = \lim_{k\to\infty}\left(\frac{1}{2}+\frac{1}{2k}\right) = \frac{1}{2} < 1 \Rightarrow \text{konvergent}$$

f)

$$a_k = \frac{k^k}{k!} \qquad \frac{a_{k+1}}{a_k} = \frac{\frac{(k+1)^{k+1}}{(k+1)!}}{\frac{k^k}{k!}} = \frac{k!}{(k+1)!}\,\frac{(k+1)^{k+1}}{k^k} = \frac{k!\quad (k+1)^k(k+1)}{k!(k+1)\quad k^k}$$

$$= \frac{(k+1)^k}{k^k} = \left(\frac{k+1}{k}\right)^k = \left(1+\frac{1}{k}\right)^k$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e > 1 \Rightarrow \text{divergent}$$

g)

$$a_k = \frac{2^k}{k!} \qquad \frac{a_{k+1}}{a_k} = \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} = \frac{k!}{(k+1)!}\,\frac{2^{k+1}}{2^k} = \frac{1}{k+1}\,2$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{2}{k+1} = 0 < 1 \Rightarrow \text{konvergent}$$

h) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2} = \sum_{k=2}^{\infty} \frac{\ln k}{k^2} = \sum_{k=1}^{\infty} \frac{\ln(k+1)}{(k+1)^2}$ $\qquad a_k = f(k) = \dfrac{\ln(k+1)}{(k+1)^2}$

$$f(x) = \frac{\ln(x+1)}{(x+1)^2} \qquad f'(x) = \frac{\frac{1}{x+1}(x+1)^2 - \ln(x+1)\,2(x+1)}{(x+1)^4}$$

$$= \frac{(x+1) - 2(x+1)\ln(x+1)}{(x+1)^4} = \frac{1 - 2\ln(x+1)}{(x+1)^3}$$

$$f'(x) = 0 \qquad 1 - 2\ln(x+1) = 0 \qquad \ln(x+1) = \frac{1}{2} \qquad x+1 = e^{\frac{1}{2}} = \sqrt{e} \qquad x = \sqrt{e} - 1 \approx 0{,}65$$

$f'(x)$		$+$		$-$	$\to x$
	-1		$\sqrt{e}-1$		

f streng monoton fallend für $x > \sqrt{e} - 1$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\ln(x+1)}{(x+1)^2}\,dx = \lim_{\mu\to\infty} \int_1^{\mu} \frac{\ln(x+1)}{(x+1)^2}\,dx \overset{\text{Subst.}}{\underset{t=x+1}{=}} \lim_{\mu\to\infty} \int_2^{\mu+1} \frac{\ln t}{t^2}\,dt$$

$$= \lim_{u\to\infty} \int_2^{u} \frac{\ln t}{t^2}\,dt = \lim_{u\to\infty} \left[-\frac{\ln t}{t} - \frac{1}{t} \right]_2^u$$

$$= \lim_{u\to\infty} \left(-\frac{\ln u}{u} - \frac{1}{u} + \frac{\ln 2}{2} + \frac{1}{2} \right) = \lim_{u\to\infty} \left(-\frac{\ln u}{u} \right) + \frac{\ln 2}{2} + \frac{1}{2}$$

$$\overset{\text{L'H.}}{=} \lim_{u\to\infty} \left(-\frac{1}{u} \right) + \frac{1}{2}(\ln 2 + 1) = \frac{1}{2}(\ln 2 + 1) \implies \text{konvergent}$$

7.3 a) $a_k = 4^k \qquad \displaystyle\lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{4^k}{4^{k+1}} = \lim_{k\to\infty} \frac{4^k}{4^k 4} = \frac{1}{4} \qquad r = \frac{1}{4}$

b) $a_k = k^4 \qquad \displaystyle\lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{k^4}{(k+1)^4} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^4 = \lim_{k\to\infty} \left(\frac{k+1-1}{k+1}\right)^4$

$$= \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right)^4 = 1 \qquad r = 1$$

c) $a_k = \dfrac{3^k}{k!} \qquad \displaystyle\lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{\frac{3^k}{k!}}{\frac{3^{k+1}}{(k+1)!}} = \lim_{k\to\infty} \frac{(k+1)!\,3^k}{k!\,3^{k+1}}$

$$= \lim_{k\to\infty} \frac{k+1}{3} = \infty$$

d)

$$a_k = \frac{k!}{k^k} \qquad \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{\frac{k!}{k^k}}{\frac{(k+1)!}{(k+1)^{k+1}}} = \lim_{k\to\infty} \frac{k!\ (k+1)^{k+1}}{(k+1)!\ k^k}$$

$$= \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)\,k^k} = \lim_{k\to\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k$$

$$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e \qquad r = e \approx 2{,}718$$

e)

$$a_k = \frac{k^2}{2^k} \qquad \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{\frac{k^2}{2^k}}{\frac{(k+1)^2}{2^{k+1}}} = \lim_{k\to\infty} \frac{k^2\ 2^{k+1}}{(k+1)^2\ 2^k}$$

$$= \lim_{k\to\infty} \frac{2k^2}{(k+1)^2} = 2 \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 = 2 \lim_{k\to\infty} \left(\frac{k+1-1}{k+1}\right)^2$$

$$= 2 \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right)^2 = 2 \qquad r = 2$$

f)

$$a_k = \left(1 - \frac{1}{2k}\right)^{k^2} = \left(1 - \frac{1}{2k}\right)^{k\cdot k} = \left(\left(1 - \frac{1}{2k}\right)^k\right)^k$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left(\left(1 - \frac{1}{2k}\right)^k\right)^k} = \lim_{k\to\infty} \left(1 - \frac{1}{2k}\right)^k$$

$$= \lim_{k\to\infty} \left(1 + \frac{-\frac{1}{2}}{k}\right)^k = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} \qquad r = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}} = \sqrt{e}$$

7.4 a)

$$f(x) = \frac{1}{x} - \ln x \qquad f(x_0) = f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} - \frac{1}{x} \qquad f'(x_0) = f'(1) = -2$$

$$f''(x) = \frac{2}{x^3} + \frac{1}{x^2} \qquad f''(x_0) = f''(1) = 3$$

$$f'''(x) = -\frac{6}{x^4} - \frac{2}{x^3} \qquad f'''(x_0) = f'''(1) = -8$$

$$f''''(x) = \frac{24}{x^5} + \frac{6}{x^4} \qquad f''''(x_0) = f''''(1) = 30$$

$$f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f'''(x_0)(x-x_0)^2 + \frac{1}{4!} f''''(x_0)(x-x_0)^4$$

$$= 1 - 2(x-1) + \frac{3}{2}(x-1)^2 - \frac{4}{3}(x-1)^3 + \frac{5}{4}(x-1)^4$$

b)

$$f(x) = \ln \cosh x \qquad f(x_0) = f(0) = 0$$

$$f'(x) = \frac{1}{\cosh x} \sinh x = \tanh x \qquad f'(x_0) = f'(0) = 0$$

$$f''(x) = 1 - \tanh^2 x \qquad f''(x_0) = f''(0) = 1$$

$$f'''(x) = -2\tanh x\,(1-\tanh^2 x) \qquad f'''(x_0) = f'''(0) = 0$$

$$= -2\tanh x + 2\tanh^3 x$$

$$f''''(x) = -2(1-\tanh^2 x) + 6\tanh^2 x\,(1-\tanh^2 x) \qquad f''''(x_0) = f''''(0) = -2$$

$$f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f'''(x_0)(x-x_0)^2 + \frac{1}{4!} f''''(x_0)(x-x_0)^4$$

$$= \frac{1}{2}x^2 - \frac{1}{12}x^4$$

c)

$$f(x) = e^{\sin x}$$

$$f'(x) = e^{\sin x} \cos x$$

$$f''(x) = e^{\sin x} \cos^2 x - e^{\sin x} \sin x$$

$$f'''(x) = e^{\sin x} \cos^3 x - 2 e^{\sin x} \cos x \sin x - e^{\sin x} \cos x \sin x - e^{\sin x} \cos x$$

$$= e^{\sin x} \cos^3 x - 3 e^{\sin x} \cos x \sin x - e^{\sin x} \cos x$$

$$= e^{\sin x} \left(\cos^3 x - 3\cos x \sin x - \cos x \right)$$

$$f''''(x) = e^{\sin x} \cos x \left(\cos^3 x - 3\cos x \sin x - \cos x \right)$$

$$+ e^{\sin x} \left(-3\cos^2 x \sin x + 3\sin^2 x - 3\cos^2 x + \sin x \right)$$

$$f(x_0) = f(\pi) = 1 \qquad f'(x_0) = f'(\pi) = -1 \qquad f''(x_0) = f''(\pi) = 1$$

$$f'''(x_0) = f'''(\pi) = 0 \qquad f''''(x_0) = f''''(\pi) = -3$$

$$f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f'''(x_0)(x-x_0)^2 + \frac{1}{4!} f''''(x_0)(x-x_0)^4$$

$$= 1 - (x-\pi) + \frac{1}{2}(x-\pi)^2 - \frac{1}{8}(x-\pi)^4$$

7.5 a)

$$f(x) = x^{-2} - 2x^{-1}$$

$$f'(x) = -2x^{-3} + 2x^{-2}$$

$$f''(x) = 2\cdot 3x^{-4} - 2\cdot 2x^{-3}$$

$$f'''(x) = -2\cdot 3\cdot 4x^{-5} + 2\cdot 2\cdot 3x^{-4}$$

$$f''''(x) = 2\cdot 3\cdot 4\cdot 5x^{-6} - 2\cdot 2\cdot 3\cdot 4x^{-5}$$

$$f^{(k)}(x) = (-1)^k (k+1)!\,x^{-(k+2)} - (-1)^k\, 2k!\,x^{-(k+1)}$$

$$f^{(k)}(x_0) = f^{(k)}(1) = (-1)^k(k+1)! - (-1)^k 2k! = (-1)^k\left((k+1)! - 2k!\right)$$

$$= (-1)^k\left(k!(k+1) - 2k!\right) = (-1)^k k!\left(k+1-2\right) = (-1)^k k!(k-1)$$

$$a_k = \frac{1}{k!} f^{(k)}(x_0) = \frac{1}{k!}(-1)^k k!(k-1) = (-1)^k(k-1)$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k = \sum_{k=0}^{\infty} a_k(x-x_0)^k = \sum_{k=0}^{\infty}(-1)^k(k-1)(x-1)^k$$

$$= -1 + (x-1)^2 - 2(x-1)^3 + 3(x-1)^4 - 4(x-1)^5 + \ldots$$

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{k-1}{k+1-1} = \lim_{k\to\infty}\frac{k-1}{k} = \lim_{k\to\infty}\left(1-\frac{1}{k}\right) = 1 \qquad r = 1$$

$x = x_0 - r = 0$ $\quad -1+1+2+3+4+\ldots$ divergent

$x = x_0 + r = 2$ $\quad -1+1-2+3-4+\ldots$ divergent

Konvergenzintervall $I =]0;2[$

b)

$$f(x) = \ln(4-x)$$

$$f'(x) = \frac{1}{4-x}(-1) = \frac{1}{x-4} = (x-4)^{-1}$$

$$f''(x) = -(x-4)^{-2}$$

$$f'''(x) = 2(x-4)^{-3}$$

$$f''''(x) = -2\cdot 3\,(x-4)^{-4}$$

$$f^{(k)}(x) = (-1)^{k+1}(k-1)!\,(x-4)^{-k}$$

$$f^{(k)}(x_0) = f^{(k)}(3) = (-1)^{k+1}(k-1)!\,(-1)^{-k} = -(k-1)!$$

$$a_k = \frac{1}{k!} f^{(k)}(x_0) = -\frac{(k-1)!}{k!} = -\frac{1}{k} \quad \text{für } k > 0 \qquad a_0 = f(x_0) = f(3) = 0$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k = \sum_{k=1}^{\infty} \left(-\frac{1}{k}\right)(x-3)^k$$

$$= -(x-3) - \frac{1}{2}(x-3)^2 - \frac{1}{3}(x-3)^3 - \frac{1}{4}(x-3)^4 - \ldots$$

$$r = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{k+1}} = \lim_{k\to\infty} \frac{k+1}{k} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right) = 1$$

$x = x_0 - r = 2 \qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ konvergent nach Leibniz-Kriterium

$x = x_0 + r = 4 \qquad -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots = -\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\right)$ divergent nach Integralkriterium

Konvergenzintervall $I = [2; 4[$

c)

$$f(x) = (3-2x)^{-1}$$

$$f'(x) = -(3-2x)^{-2}(-2) = (3-2x)^{-2}2$$

$$f''(x) = -2(3-2x)^{-3}2(-2) = 2(3-2x)^{-3}2^2$$

$$f'''(x) = -2\cdot 3(3-2x)^{-4}2^2(-2) = 2\cdot 3(3-2x)^{-4}2^3$$

$$f''''(x) = -2\cdot 3\cdot 4(3-2x)^{-5}2^3(-2) = 2\cdot 3\cdot 4(3-2x)^{-5}2^4$$

$$f^{(k)}(x) = k!(3-2x)^{-(k+1)}2^k$$

$$f^{(k)}(x_0) = f^{(k)}(1) = k!\,2^k$$

$$a_k = \frac{1}{k!}f^{(k)}(x_0) = 2^k$$

$$\sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k = \sum_{k=0}^{\infty}2^k(x-1)^k = 1+2(x-1)+4(x-1)^2+8(x-1)^3+\ldots$$

$$r = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{2^k}{2^{k+1}} = \lim_{k\to\infty}\frac{1}{2} = \frac{1}{2}$$

$$x = x_0 - r = \frac{1}{2}$$

$$1+2\left(-\frac{1}{2}\right)+2^2\left(-\frac{1}{2}\right)^2+2^3\left(-\frac{1}{2}\right)^3+\ldots = 1-1+1-1+\ldots \quad \text{divergent}$$

$$x = x_0 + r = \frac{3}{2}$$

$$1+2\cdot\frac{1}{2}+2^2\left(\frac{1}{2}\right)^2+2^3\left(\frac{1}{2}\right)^3+\ldots = 1+1+1+1+\ldots \quad \text{divergent}$$

Konvergenzintervall $I =]\frac{1}{2};\frac{3}{2}[$

$$\sum_{k=0}^{\infty}2^k(x-1)^k = \sum_{k=0}^{\infty}q^k$$ geometrische Reihe mit $q = 2(x-1)$

Im Konvergenzintervall ist $|x-1| < \frac{1}{2}$ und damit $|q| = |2(x-1)| < 1$

$$\sum_{k=0}^{\infty}q^k = \frac{1}{1-q} = \frac{1}{1-2(x-1)} = \frac{1}{3-2x}$$

7.6 a)

$$\frac{\sin x}{x} = \frac{x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\ldots}{x} = 1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\ldots$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\ldots\right) = 1$$

b)

$$x - \ln(1+x) = x - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right)$$
$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \ldots = x^2\left(\frac{1}{2} - \frac{1}{3}x + \frac{1}{4}x^2 - \ldots\right)$$

$$\sin(x^2) = x^2 - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \ldots = x^2\left(1 - \frac{1}{3!}(x^2)^2 + \frac{1}{5!}(x^2)^4 - \ldots\right)$$

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{\sin(x^2)} = \lim_{x\to 0} \frac{\frac{1}{2} - \frac{1}{3}x + \frac{1}{4}x^2 - \ldots}{1 - \frac{1}{3!}(x^2)^2 + \frac{1}{5!}(x^2)^4 - \ldots} = \frac{1}{2}$$

c)

$$(e^x - 1)^2 = \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots - 1\right)^2 = \left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\right)^2$$
$$= \left(x\left(1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \ldots\right)\right)^2 = x^2\left(1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \ldots\right)^2$$

$$\lim_{x\to 0} \frac{x^2 e^x}{(e^x - 1)^2} = \lim_{x\to 0} \frac{e^x}{\left(1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \ldots\right)^2} = 1$$

7.7 a)

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \approx 1 + x + \frac{1}{2!}x^2$$

$$\to \quad 1 + x + \frac{1}{2}x^2 = -x^2 + \frac{3}{2} \qquad \frac{3}{2}x^2 + x - \frac{1}{2} = 0 \qquad 3x^2 + 2x - 1 = 0$$

$$x_{1;2} = \frac{1}{6}\left(-2 \pm \sqrt{16}\right) \qquad x_1 = -1 \qquad x_2 = \frac{1}{3}$$

Tangentenverfahren von Newton: $x_1 = -1{,}076699$ $\quad x_2 = 0{,}330064$

b)

$$f(x) = \ln(x-1) \qquad f'(x) = \frac{1}{x-1} \qquad f''(x) = -\frac{1}{(x-1)^2}$$

$$f(x_0) = f(2) = 0 \qquad f'(x_0) = f'(2) = 1 \qquad f''(x_0) = f''(2) = -1$$

$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 = x - 2 - \frac{1}{2}(x-2)^2$$

$$\to \quad x - 2 - \frac{1}{2}(x-2)^2 = -\frac{1}{2}x^2 + 3 \qquad x = \frac{7}{3}$$

Tangentenverfahren von Newton: $x = 2{,}33012$

7.8

$$f(x) = a_0 + a_1 x + a_2 x^2$$
$$f'(x) = a_1 + 2a_2 x$$
$$f''(x) = 2a_2$$
$$f'''(x) = 0$$
$$f^{(k)}(x) = 0 \quad \text{für } k>2$$
$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$
$$= a_0 + a_1 x_0 + a_2 x_0^2 + (a_1 + 2a_2 x_0)(x-x_0) + \frac{1}{2} 2a_2 (x-x_0)^2$$
$$= a_0 + a_1 x + a_2 x^2$$

Für jeden Entwicklungspunkt erhält man das gleiche Ergebnis, die ursprüngliche Funktion

7.9 a)

Für $x \neq 0$ gilt $f(x) = x^3 \ln(x^2)$ und
$$f'(x) = 3x^2 \ln(x^2) + x^3 \frac{1}{x^2} 2x = 3x^2 \ln(x^2) + 2x^2$$
$$f''(x) = 6x \ln(x^2) + 3x^2 \frac{1}{x^2} 2x + 4x = 6x \ln(x^2) + 10x$$

Für $x=0$ gilt $f(0) = 0$ und
$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^3 \ln(x^2)}{x} = \lim_{x\to 0} x^2 \ln(x^2) = \lim_{x\to 0} \frac{\ln(x^2)}{\frac{1}{x^2}}$$
$$\overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{\frac{1}{x^2} 2x}{-\frac{2}{x^3}} = \lim_{x\to 0}(-x^2) = 0$$
$$f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} \frac{3x^2 \ln(x^2) + 2x^2}{x} = \lim_{x\to 0}(3x \ln(x^2) + 2x)$$
$$= 3 \lim_{x\to 0}(x \ln(x^2)) = 3 \lim_{x\to 0} \frac{\ln(x^2)}{\frac{1}{x}} \overset{\text{L'H.}}{=} 3 \lim_{x\to 0} \frac{\frac{1}{x^2} 2x}{-\frac{1}{x^2}} = 3 \lim_{x\to 0}(-2x) = 0$$

Für $x_0 = 0$ gilt $f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2$
$$= f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 = 0$$

b)

$$f'''(0) = \lim_{x\to 0} \frac{f''(x)-f''(0)}{x-0} = \lim_{x\to 0} \frac{6x \ln(x^2) + 10x}{x} = \lim_{x\to 0}(6 \ln(x^2) + 10) = -\infty$$

$f^{(k)}(x_0) = f^{(k)}(0)$ existiert nicht für $k \geq 3$

keine Taylorentwicklung mit $x_0 = 0$ möglich

7.10 a) $f^{(k)}(x_0) = f^{(k)}(0) = 0$ für alle $k \in \mathbb{N}$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)x^k = 0$$

b) Taylorreihe stimmt für $x \neq 0$ nicht mit $f(x)$ überein.

7.11

Für $|x| \leq 1$ gilt $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$

$$\frac{\arctan x}{x} = 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \ldots$$

für $|x| \leq 1$ und $x \neq 0$

$$\frac{\arctan x}{x} \approx 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6$$

$$\int_0^{\frac{1}{2}} \frac{\arctan x}{x}\,dx = \lim_{\varepsilon \to 0} \int_\varepsilon^{\frac{1}{2}} \frac{\arctan x}{x}\,dx \approx \lim_{\varepsilon \to 0} \int_\varepsilon^{\frac{1}{2}} \left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6\right)dx$$

$$= \lim_{\varepsilon \to 0} \left[x - \frac{1}{9}x^3 + \frac{1}{25}x^5 - \frac{1}{49}x^7\right]_\varepsilon^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{9}\left(\frac{1}{2}\right)^3 + \frac{1}{25}\left(\frac{1}{2}\right)^5 - \frac{1}{49}\left(\frac{1}{2}\right)^7 = 0{,}48720$$

7.12

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \ldots \quad \text{für } -1 \leq x < 1$$

$$\ln\left(1 - \frac{k}{m}\frac{x}{v_{x0}}\right) = -\frac{kx}{mv_{x0}} - \frac{1}{2}\left(\frac{kx}{mv_{x0}}\right)^2 - \frac{1}{3}\left(\frac{kx}{mv_{x0}}\right)^3 - \frac{1}{4}\left(\frac{kx}{mv_{x0}}\right)^4 - \ldots$$

$$y = f(x) = \left(v_{y0} + g\frac{m}{k}\right)\frac{x}{v_{x0}} + g\left(\frac{m}{k}\right)^2 \ln\left(1 - \frac{kx}{mv_{x0}}\right)$$

$$= \frac{v_{y0}}{v_{x0}}x + g\frac{m}{k}\frac{x}{v_{x0}} + g\left(\frac{m}{k}\right)^2\left(-\frac{kx}{mv_{x0}} - \frac{1}{2}\left(\frac{kx}{mv_{x0}}\right)^2 - \frac{1}{3}\left(\frac{kx}{mv_{x0}}\right)^3 - \frac{1}{4}\left(\frac{kx}{mv_{x0}}\right)^4 - \ldots\right)$$

$$= \frac{v_{y0}}{v_{x0}}x - \frac{g}{2v_{x0}^2}x^2 - g\left(\frac{m}{k}\right)^2\left(\frac{1}{3}\left(\frac{kx}{mv_{x0}}\right)^3 + \frac{1}{4}\left(\frac{kx}{mv_{x0}}\right)^4 + \ldots\right)$$

$$= \frac{v_{y0}}{v_{x0}}x - \frac{g}{2v_{x0}^2}x^2 - g\left(\frac{1}{3}\frac{k}{m}\left(\frac{x}{v_{x0}}\right)^3 + \frac{1}{4}\left(\frac{k}{m}\right)^2\left(\frac{x}{v_{x0}}\right)^4 + \ldots\right)$$

$$\xrightarrow{k \to 0} \frac{v_{y0}}{v_{x0}}x - \frac{g}{2v_{x0}^2}x^2 \qquad \text{Wurfparabel}$$

7.13

$$s_k = s_1 e^{-\delta \frac{2\pi}{\omega}(k-1)} = s_1 \left(e^{-\delta \frac{2\pi}{\omega}}\right)^{k-1} = s_1 q^{k-1} \quad \text{mit} \quad q = e^{-\delta \frac{2\pi}{\omega}} < 1$$

$$s = \sum_{k=1}^{\infty} s_k = s_1 \sum_{k=1}^{\infty} q^{k-1} = s_1 (1 + q + q^2 + \dots) = s_1 \frac{1}{1-q} = s_1 \frac{1}{1 - e^{-\delta \frac{2\pi}{\omega}}}$$

$\Rightarrow$ endliche Strecke

7.14 a)

f ungerade $\Rightarrow$ $a_n = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx = \frac{2}{\pi} \left[-\frac{1}{n} \cos(nx) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(-\frac{1}{n} \cos(n\pi) + \frac{1}{n} \right) = \frac{2}{\pi} \frac{1}{n} \left(1 - (-1)^n \right)$$

n gerade: $b_n = 0$ $\quad$ n ungerade: $b_n = \frac{4}{\pi} \frac{1}{n}$

$$\sum_{n=1}^{\infty} b_n \sin(nx) = \frac{4}{\pi} \left(\sin x + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots \right)$$

b)

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} -x + \pi \, dx = \frac{1}{\pi} \left[-\frac{1}{2} x^2 + \pi x \right]_0^{\pi} = \frac{1}{\pi} \frac{1}{2} \pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \int_0^{\pi} (-x + \pi) \cos(nx)\, dx$$

$$= -\frac{1}{\pi} \int_0^{\pi} x \cos(nx)\, dx + \int_0^{\pi} \cos(nx)\, dx$$

$$= -\frac{1}{\pi} \left[\frac{\cos(nx)}{n^2} + \frac{x \sin(nx)}{n} \right]_0^{\pi} + \left[\frac{1}{n} \sin(nx) \right]_0^{\pi}$$

$$= -\frac{1}{\pi} \left(\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} \right) = \frac{1}{\pi} \frac{1}{n^2} \left(1 - \cos(n\pi) \right) = \frac{1}{\pi} \frac{1}{n^2} \left(1 - (-1)^n \right)$$

n gerade: $a_n = 0$ $\quad$ n ungerade: $a_n = \frac{2}{\pi} \frac{1}{n^2}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}(-x+\pi)\sin(nx)\,dx$$

$$= -\frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx + \int_0^{\pi}\sin(nx)\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right]_0^{\pi} + \left[-\frac{1}{n}\cos(nx)\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left(-\frac{\pi\cos(n\pi)}{n}\right) + \left(-\frac{1}{n}\cos(n\pi) + \frac{1}{n}\right) = \frac{1}{n}$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left(\cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \dots\right) + \sin x + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \dots$$

c)

f gerade $\Rightarrow b_n = 0$ $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} = \pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2} + \frac{x\sin(nx)}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2}\right) = -\frac{2}{\pi}\frac{1}{n^2}\left(1 - \cos(n\pi)\right) = -\frac{2}{\pi}\frac{1}{n^2}\left(1-(-1)^n\right)$$

n gerade: $a_n = 0$ $\quad$ n ungerade: $a_n = -\frac{4}{\pi}\frac{1}{n^2}$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \dots\right)$$

d)

$$f \text{ gerade} \Rightarrow b_n = 0 \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{4}{\pi}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\cos x\,dx = \frac{2}{\pi}\left[\frac{1}{2}\sin^2 x\right]_0^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\Big(\sin((1-n)x) + \sin((1+n)x)\Big)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin((1-n)x)\,dx + \frac{1}{\pi}\int_0^{\pi}\sin((1+n)x)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{1-n}\cos((1-n)x)\right]_0^{\pi} + \frac{1}{\pi}\left[-\frac{1}{1+n}\cos((1+n)x)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(-\frac{1}{1-n}\cos((1-n)\pi) + \frac{1}{1-n} - \frac{1}{1+n}\cos((1+n)\pi) + \frac{1}{1+n}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n-1}\cos((n-1)\pi) - \frac{1}{n+1}\cos((n+1)\pi) - \frac{1}{n-1} + \frac{1}{n+1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n-1}(-1)^{n-1} - \frac{1}{n+1}(-1)^{n+1} - \frac{1}{n-1} + \frac{1}{n+1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n-1}(-1)^{n+1} - \frac{1}{n+1}(-1)^{n+1} - \frac{1}{n-1} + \frac{1}{n+1}\right)$$

$$= \frac{1}{\pi}\left((-1)^{n+1}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) - \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)\left((-1)^{n+1} - 1\right) = \frac{2}{\pi}\,\frac{1}{(n-1)(n+1)}\left((-1)^{n+1} - 1\right)$$

$$n \text{ ungerade}: \ a_n = 0 \qquad n \text{ gerade}: \ a_n = -\frac{4}{\pi}\,\frac{1}{(n-1)(n+1)}$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{1\cdot 3} + \frac{\cos(4x)}{3\cdot 5} + \frac{\cos(6x)}{5\cdot 7} + \ldots\right)$$

e)

$$f \text{ gerade} \Rightarrow b_n = 0 \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{2x}{n^2}\cos(nx) + \left(\frac{x^2}{n} - \frac{2}{n^3}\right)\sin(nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\,\frac{2\pi}{n^2}\cos(n\pi) = 4(-1)^n\frac{1}{n^2}$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = \frac{\pi^2}{3} + 4\left(-\cos x + \frac{\cos(2x)}{2^2} - \frac{\cos(3x)}{3^2} + \frac{\cos(4x)}{4^2} - \ldots\right)$$